Some inequalities for the Tutte polynomial

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Abstract

We prove that the Tutte polynomial of a coloopless paving matroid is convex along the portions of the line segments $x + y = p$ lying in the positive quadrant. Every coloopless paving matroids is in the class of matroids which contain two disjoint bases or whose ground set is the union of two bases of M^*. For this latter class we give a proof that $T_M(a, a) \leq \max\{T_M(2a, 0), T_M(0, 2a)\}$ for $a \geq 2$. We conjecture that $T_M(1, 1) \leq \max\{T_M(2, 0), T_M(0, 2)\}$ for the same class of matroids. We also prove this conjecture for some families of graphs and matroids.

1 Introduction

The Tutte polynomial is a two variable polynomial which can be defined for a graph G or, more generally, a matroid M. The Tutte polynomial has many interesting combinatorial interpretations when evaluated at different points (x, y) and along several algebraic curves. For example, for a graph G, the Tutte polynomial along the line $y = 0$ is the chromatic polynomial, after a suitable change of variable and multiplication by an easy term. Similarly, we can get the flow polynomial of a graph and the all terminal reliability of a network and the partition function of the Q-state Potts model. When considering a GF(q)-representable matroid, the Tutte polynomial gives us the weight enumerator of linear codes over GF(q) associated to M. All the necessary background on the Tutte polynomial is contained in Section 2.

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It is well-known [3] that the Tutte polynomial of a matroid M has an expansion

$$T_M(x,y) = \sum_{i,j} t_{ij} x^i y^j,$$

in which each coefficient t_{ij} is non-negative. Consequently, for $m \geq 0$ and for any b, if (x,y) lies on the portion of the line $y = mx + b$ lying in the positive quadrant, then T_M increases as x increases. The simplicity of the behaviour of T along lines with positive gradient suggests the study of the behaviour of T_M along lines with negative gradient in the positive quadrant. Merino and Welsh [19] were the first to consider this and were particularly interested in resolving the question of whether the Tutte polynomial is convex along the portion of the line $x + y = 2$ lying in the positive quadrant. They made the following intriguing conjecture.

Conjecture 1.1. *Let G be a 2-connected graph with no loops. Then*

$$\max\{T_G(2,0), T_G(0,2)\} \geq T_G(1,1). \tag{1}$$

Any graph with at least one loop and at least one isthmus fails to satisfy (1), so (1) cannot hold for all graphs. The main reason for the particular interest in the points $(2,0)$, $(0,2)$ and $(1,1)$ is that in a connected graph G, $T_G(2,0)$, $T_G(0,2)$ and $T_G(1,1)$ give the number of acyclic orientations, totally cyclic orientations and spanning trees in G. Definitions of acyclic and totally cyclic orientations are contained in Section 2.

A related question is to determine whether any loopless 2-connected graph G satisfies the apparently stronger requirement

$$T_G(2,0)T_G(0,2) \geq (T_G(1,1))^2.$$

Relatively little progress has been made to resolve these questions. However, Jackson in [15] has shown, with a clever argument, that for any connected matroid M,

$$T_G(3,0)T_G(0,3) \geq (T_G(1,1))^2.$$

In this paper we make three contributions. First, in Section 4, we show that the Tutte polynomial T of a coloopless paving matroid satisfies the inequality

$$tT(x_1,y_1) + (1-t)T(x_2,y_2) \geq T(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2), \tag{2}$$

where $0 \leq t \leq 1$ and x_1, x_2, y_1, y_2 are non-negative and satisfy $x_1 + y_1 = x_2 + y_2$. That is, T is convex along the portions of the line segments $x + y = p$ lying in the positive quadrant. A paving matroid is one in which all circuits have size at least $r(M)$. Interest in them stems from a conjecture in [16] which says that asymptotically almost every matroids is paving. The special case of (2), obtained by setting $x_1 = y_2 = 2$, $x_2 = y_1 = 0$ and $t = 1/2$, establishes (1) for the class of paving matroids. Therefore if the above conjecture is true then we have established (1) for, asymptotically, almost all coloopless matroids.

Second, in Section 5, we prove that (1) holds for some smaller classes of matroids and graphs that are not paving matroids. Finally, in Section 3, we prove that if the ground set of M contains two disjoint bases then $T_M(0, 2a) \geq T_M(a, a)$ and dually if the ground set of M is the union of two bases then $T_M(2a, 0) \geq T_M(a, a)$. These results cannot be obtained with the methods used by Jackson in [15].

We conclude with a brief discussion of the natural question of for which matroids is T_M a convex function in the positive quadrant?

2 Preliminaries

We assume that the reader has some familiarity with matroid and graph theory. For matroid theory we follow Oxley's book [22] and for graph theory we follow Diestel's book [9].

The Tutte polynomial is a matroid invariant over the ring $\mathbb{Z}[x, y]$. Further details of many of the concepts treated here can be found in Welsh [27] and Oxley and Brylawski [6].

Some of the richness of the Tutte polynomial is due to its numerous equivalent definitions. One of the simplest definitions, which is often the easiest way to prove properties of the Tutte polynomial, uses the notion of rank.

If $M = (E, r)$ is a matroid, where r is the rank-function of M, and $A \subseteq E$, we denote $r(E) - r(A)$ by $z(A)$ and $|A| - r(A)$ by $n(A)$.

Definition 2.1. *The Tutte polynomial of M, $T_M(x, y)$, is defined as follows:*

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{z(A)} (y-1)^{n(A)} \,. \tag{3}$$

Almost immediately we see that $T_M(1, 1)$ equals the number of bases of M and $T_M(2, 2)$ equals $2^{|E|}$. Recall that if $M = (E, r)$ is a matroid, then $M^* = (E, r^*)$ is its dual matroid, where $r^*(A) = |A| - r(E) + r(E \setminus A)$. Because $z_{M^*}(A) = n_M(E \setminus A)$ and $n_{M^*}(A) = z_M(E \setminus A)$ it follows that $T_M(x, y) = T_{M^*}(y, x)$.

For a graphic matroid $M(G)$, the evaluations of the Tutte polynomial at $(2, 0)$ and $(0, 2)$ equal the number of acyclic orientations and the number of totally cyclic orientations of G, respectively. An acyclic orientation of a graph G is an orientation where there are no directed cycles. A totally cyclic orientation is an orientation where every edge is in a directed cycle. See [6] for a proof of this result. In this situation we let $\alpha(G)$ and $\alpha^*(G)$ denote $T_G(2, 0)$ and $T_G(0, 2)$ respectively. If G is connected, the number of spanning trees of G is the evaluation of the Tutte polynomial at $(1, 1)$ and this quantity is denoted by $\tau(G)$.

The Tutte polynomial may be also defined by a linear recursion relation given by deleting and contracting elements that are neither loops nor isthmuses.

Definition 2.2. *If M is a matroid, and e is an element that is neither an isthmus nor a loop, then*

$$T_M(x,y) = T_{M\setminus e}(x,y) + T_{M/e}(x,y). \tag{4}$$

If there is no such element e, then $T_M(x,y) = x^i y^j$ where i and j are the number of isthmuses and loops of M respectively.

The proof that Definition 2.1 and 2.2 are equivalent can be found in [6]. We still require another (equivalent) definition of the Tutte polynomial but first we introduce the relevant notions.

Let us fix an ordering $\prec$ on the elements of M, say $E = \{e_1, \ldots, e_m\}$, where $e_i \prec e_j$ if $i < j$. Given a fixed basis S, an element e is called *internally active* if $e \in S$ and it is the smallest edge with respect to $\prec$ in the only cocircuit disjoint from $S \setminus \{e\}$. Dually, an element f is *externally active* if $f \notin S$ and it is the smallest element in the only circuit contained in $S \cup \{f\}$. We define t_{ij} to be the number of bases with i internally activity elements and j externally activity elements. In [25] Tutte defined T_M using these concepts. A proof of the equivalence with Definition 2.1 can be found in [3].

Definition 2.3. *If $M = (E, r)$ is a matroid with a total order on its ground set, then*

$$T_M(x,y) = \sum_{i,j} t_{ij} x^i y^j \ . \tag{5}$$

In particular, the coefficients t_{ij} are independent of the total order used on the ground set.

By an inductive argument using equation (4), it can be proved that $t_{10} = t_{01}$ when $E(M) \geq 2$. This is one of a number of identities known to hold for the coefficients t_{ij}. For a complete characterization of all the affine linear relations that hold among the coefficients t_{ij} see Theorem 6.2.13 in [6]. From there we extract the relations that we need.

Theorem 2.4. *If a rank-r matroid M with m elements has neither loops nor isthmuses, then*

(a) $t_{ij} = 0$, whenever $i > r$ or $j > m - r$;

(b) $t_{r0} = 1$ and $t_{0,m-r} = 1$;

(c) $t_{rj} = 0$ for all $j > 0$ and $t_{i,m-r} = 0$ for all $i > 0$.

The previous result follows easily from Definition 2.3. In [6] the statement is for simple matroids (geometries) but it is easy to extend it to matroids with parallel elements.

3 Some inequalities for the Tutte polynomial

From the results in the previous section it is easy to prove the following result stated in [18].

Theorem 3.1. *If a matroid M has neither loops nor isthmuses, then*

$$\max\{T_M(4,0), T_M(0,4)\} \geq T_M(2,2).$$

Proof. Let r be the rank and m the number of elements of M.

$$\begin{aligned} \max\{T_M(4,0), T_M(0,4)\} &\geq \max\{4^r, 4^{m-r}\} \\ &= \max\{2^{2r}, 2^{2(m-r)}\} \\ &\geq 2^m = T_M(2,2)\ , \end{aligned}$$

where the first inequality follows from equation (5) combined with (b) in Theorem 2.4. $\square$

Note that, for a matroid $M = (E, r)$ with dual $M^* = (E, r^*)$, the following inequalities are equivalent for any $A \subseteq E$.

$$|A| \leq |E| - 2(r(E) - r(A)), \tag{6}$$

$$|E \setminus A| \leq 2r^*(E \setminus A) \text{ and} \tag{7}$$

$$z(A) + n(A) \leq |E| - r. \tag{8}$$

We now restrict attention to matroids M in which all subsets A of the ground set E satisfy the (equivalent) inequalities above. By a classical result of J. Edmonds [10], these are the matroids that contain two disjoint bases; by duality, these are the matroids M whose ground set is the union of two bases of M^*.

As every term $(x-1)^{z(A)}(y-1)^{n(A)}$ in T_M has $x^{z(A)}y^{n(A)}$ as its monomial of maximum degree, the following theorem follows directly from the set of inequalities above.

Theorem 3.2. *If a matroid M contains two disjoint bases, then $t_{ij} = 0$, for all i and j such that $i + j > m - r$. Dually, if its ground set is the union of two bases, then $t_{ij} = 0$, for all i and j such that $i + j > r$.*

Now, it is easy to prove an infinite set of inequalities for the Tutte polynomial of a matroid that contains two disjoint bases or whose ground set is the union of two bases. This theorem was stated in [18].

Theorem 3.3. *If a matroid M contains two disjoint bases, then*

$$T_M(0, 2a) \geq T_M(a, a)\ , \tag{9}$$

for all $a \geq 2$. Dually, if its ground set is the union of two bases, then

$$T_M(2a, 0) \geq T_M(a, a), \tag{10}$$

for all $a \geq 2$.

Proof. Let us consider just the case when M has two disjoint bases: the other case follows from duality. In this situation $m - r \geq r$. From the proof of Theorem 3.1 and equation (5) we have $4^{m-r} \geq T_M(2,2) = \sum_{i,j} t_{ij} 2^{i+j}$. Multiplying this inequality by $(a/2)^{m-r}$ we get

$$(2a)^{m-r} \geq \sum_{i,j} t_{ij} \left(\frac{a}{2}\right)^{m-r} 2^{i+j} \geq \sum_{i,j} t_{ij} \left(\frac{a}{2}\right)^{i+j} 2^{i+j} = \sum_{i,j} t_{ij} a^{i+j}.$$

The second inequality follows from Theorem 3.2. Thus

$$T_M(0,2a) \geq (2a)^{m-r} \geq \sum_{i,j} t_{ij} a^{i+j} = T_M(a,a).$$

□

We can sum up the previous result by saying that if M contains two disjoint bases or its ground set is the union of two bases then

$$\max\{T_M(2a,0), T_M(0,2a)\} \geq T_M(a,a), \tag{11}$$

for $a \geq 2$. Some classes of matroids which contain two disjoint bases or whose ground set is the union of two bases are mentioned in the following

Corollary 3.4. *For a matroid M, we have that T_M satisfies (11), for all $a \geq 2$ whenever M is one of the following:*

- *an identically self-dual matroid M,*
- *a rank-r projective geometry over $GF(q)$ or its dual, for $r \geq 2$.*

Proof. A matroid $M = (E, r)$ is identically self-dual if $M = M^*$, so, B is a basis of M if and only if $E - B$ is a basis of M.

The matroid PG(r, q) contains the graphic matroid W_{r+1}, the $r+1$-wheel, as a submatroid for $r \geq 3$, see [22]. The latter contains two disjoint bases. Thus, PG(r, q) contains two disjoint bases. A projective plane of order $m \geq 4$ contains $U_{2,4} \oplus_2 U_{2,4}$ as a submatroid. Again, the latter contains two disjoint bases. Thus, such a projective plane contains two disjoint bases. The only projective plane of order 3 is the Fano matroid which clearly contains two disjoint bases. □

There are more classes of matroids that can be added to the previous list, for instance, coloopless paving matroids. However, in the next section we will prove a much stronger result for them. The graphic matroids corresponding to the families of graphs in our next result may also be added to the list.

Corollary 3.5. *For a graph G, T_G satisfies (11), for all $a \geq 2$ whenever G is one of the following:*

- *a 4-edge-connected graph,*
- *a 2-connected threshold graph,*

- *a complete bipartite graph,*
- *a series-parallel graph,*
- *a 3-regular graph,*
- *a bipartite planar graph,*
- *a Laman graph,*
- *a triangulation,*
- *the wheel graph W_n, for $n \geq 2$,*
- *the square lattice L_n, for $n \geq 2$,*
- *the n-cycle $n \geq 2$,*
- *a tree with n edges, for $n \geq 1$.*

Proof. By the classical result in [26] every 4-edge-connected graph has two edge-disjoint spanning trees. It is easy to see that 2-connected threshold and wheel graphs have two edge-disjoint spanning trees. Using the expression for computing the arboricity of a graph given in [21] we get that series-parallel, 3-regular, bipartite planar, and Laman graphs all have arboricity two, which is equivalent to having two spanning trees that cover all the edges of the graph. Triangulations are geometric duals of 3-regular planar graphs, so they have two edge-disjoint spanning trees.

It is easy to see that each of $K_{2,m}$ for $m \geq 2$, $K_{3,3}$, the square lattice L_n for $n \geq 2$, the n-cycle for $n \geq 2$, and a tree have two spanning trees which cover all the edges in the graph. With the exception of the case $n = m = 3$, if both n and m are at least 3, then $K_{n,m}$ always has two edge-disjoint spanning trees. □

4 Paving matroids

A paving matroid $M = (E, r)$ is a matroid whose circuits all have size at least r. Paving matroids are closed under minors and the set of excluded minors for the class consists of the matroid $U_{2,2} \oplus U_{0,1}$, see for example [13]. The interest about paving matroids goes back to 1976 when Dominic Welsh ask if most matroids are paving, see [22]. More recently, the authors in [16] pose as a conjecture that asymptotically almost every matroid is paving.

First, we prove that most paving matroids either contain two disjoint bases or their ground set is the union of two bases. Consequently paving matroids fall within the class of matroids considered in the previous section.

Theorem 4.1. *Let $M = (E, r)$ be a rank-r paving matroid with n elements,*

- *if $2r > n$, then E is the union of two bases,*
- *if $2r \leq n$ and M is coloopless, then M contains two disjoint bases.*

Proof. In the first case, take B_1 to be a basis of M, then $I_2 = E \setminus B_1$ has size $n - r < r$, so it is independent and we can extend it to a basis B_2. Thus $E = B_1 \cup B_2$.

In the second case, if M has a circuit C of size $r+1$, then $C' = E \setminus C$ has size $n - r - 1 \geq r - 1$. Let I be a set of size $r-1$ contained in C'. As I is independent and C is spanning, there exists $a \in C \setminus I$ such that $I \cup \{a\}$ is a basis. But $C \setminus \{a\}$ is also a basis. Thus, we have two disjoint bases.

Let M be a coloopless paving matroid with no circuits of size $r+1$ and suppose that $2r \leq n$. Let B be a basis of M. Then either $E \setminus B$ contains a basis, in which case we have finished the proof, or $r(E \setminus B) = r - 1$. In the latter case, let H be the hyperplane defined as the closure of $E \setminus B$, and $I = E \setminus H \subseteq B$. The set I has size $p+1$ with $p \geq 1$ as M is coloopless.

We show that in this case M also has two disjoint bases. Let $I' = I \setminus \{a\}$, for some $a \in I$. Then, I' is a non-empty independent set of size p with the property that for any circuit C of size r contained in H, $I' \cup C$ contains a basis of M. Thus, there is a basis B_1 of M of the form $I' \cup A_1$ for some subset A_1 of H of size $r - p$. Now, let $B_2 = \{a\} \cup A_2$ for some $A_2 \subseteq H \setminus A_1$ of size $r-1$. This is possible as $|H \setminus A_1| = (n-p-1) - (r-p) = (n-r) - 1 \geq r - 1$. Thus, B_1 and B_2 are disjoint bases of M. □

The main goal of this section is prove that for any coloopless paving matroid

$$tT(x_1, y_1) + (1-t)T(x_2, y_2) \geq T(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2), \qquad (12)$$

whenever $0 \leq t \leq 1$ and x_1, x_2, y_1, y_2 are non-negative and satisfy $x_1 + y_1 = x_2 + y_2$. Notice that this inequality is a much stronger statement than (10) as it says that T is a convex function along the portions of the line $x + y = p$ lying in the positive quadrant, rather than merely saying that the value of T at one of the endpoints of the line segment is greater than the value of T at its midpoint.

Our main tools for establishing the convexity of T are the following easy results.

Lemma 4.2. *Let M be a matroid. Either, both $T_M(x, y)$ and $T_{M^*}(x, y)$ are convex along the portion of the line $x + y = p$ lying in the positive quadrant or neither is.*

Proof. This follows directly from the equality $T_M(x, y) = T_{M^*}(y, x)$. □

Lemma 4.3. *Let M be a matroid and e in M be neither a loop nor a coloop. If $T_{M \setminus e}$ and $T_{M/e}$ are both convex along the portion of the line $x + y = p$ lying in the positive quadrant, then T_M is also convex on the same domain.*

Proof. This follows directly from the deletion-contraction formula (4) and that the sum of convex functions is also a convex function. □

The following three results deal with the convexity of T for some coloopless paving matroids. We use these cases as bases for an inductive argument later on.

Lemma 4.4. *If M is isomorphic to the paving matroid $U_{1,k+1} \oplus U_{0,l}$, where $l \geq 0$ and $k \geq 1$, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. We have

$$T_M(x,y) = y^l(y^k + \cdots + y + x) = py^l + \sum_{m=l+2}^{l+k} y^m.$$

Since y^m is convex for all $m \geq 0$ in the given region and the sum of convex functions is convex, the result follows. $\square$

Lemma 4.5. *The Tutte polynomial T_M is a convex function in the positive quadrant when M is a uniform matroid. In particular, T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant*

Proof. The Tutte polynomial of a uniform matroid can be computed easily using (3).

$$T_{U_{r,n}}(x,y) = \sum_{i=0}^{r-1} \binom{n}{i}(x-1)^{r-i} + \binom{n}{r} + \sum_{i=r+1}^{n} \binom{n}{i}(y-1)^{i-r}.$$

This can be expanded into the following expression, which may also be established directly using (5).

$$T_{U_{r,n}}(x,y) = \sum_{j=1}^{n-r} \binom{n-j-1}{r-1} y^j + \sum_{i=1}^{r} \binom{n-i-1}{n-r-1} x^i,$$

when $0 < r < n$, while $T_{U_{n,n}}(x,y) = x^n$ and $T_{U_{0,n}}(x,y) = y^n$.

As each term is a convex function we get the result. $\square$

Theorem 4.6. *If M is a rank-2 loopless and coloopless matroid, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. If M is isomorphic to the uniform matroid $U_{2,n}$, the result follows from applying the previous lemma. Otherwise, M is isomorphic to a matroid with parallel elements whose simplification is isomorphic to $U_{2,n}$.

If $n \geq 3$ or there is a parallel class of size at least 3, we can choose an element e in a non-trivial parallel class of M such that $M \setminus e$ does not have a coloop. In this case M/e is isomorphic to $U_{1,k+1} \oplus U_{0,l}$, where $l \geq 1$ and $k \geq 1$ and $M \setminus e$ is a rank-2 loopless and coloopless matroid. The result follows from Lemma 4.4, induction and Lemma 4.3.

In the last case, the simplification of M is isomorphic to $U_{2,2}$ and every element is in a parallel class of size 2. Then M is isomorphic to $U_{1,2} \oplus U_{1,2}$. Then, $T_M = (x+y)^2$ which is convex (in fact is constant) along $x + y = p$ for $p > 0$ and $0 \leq y \leq p$. $\square$

In order to establish or main result, we need the following structural result about cooopless paving matroids.

Lemma 4.7. *Let M be a rank-r coloopless paving matroid. If for every element e of M, $M \setminus e$ has a coloop, then one of the following three cases happens.*

(a) M is isomorphic to $U_{r,r+1}$.

(b) M is the 2-stretching of a uniform matroid N and N is isomorphic to $U_{s,s+1}$ or $U_{s,s+2}$, for some $s \geq 1$.

(c) M is isomorphic to $U_{1,2} \oplus U_{1,2}$.

Proof. If e is such that $M \setminus e$ has a coloop f, then $\{e, f\}$ are in either a series or form a parallel class. If there is a parallel class in a paving matroid, its rank is either 1 or 2. Thus, if $\{e, f\}$ are in a parallel class, M is isomorphic to $U_{1,2} \oplus U_{1,2}$ or $U_{1,2}$.

We can assume that M contains no non-trivial parallel classes. Hence every element belongs to a series class of size at least two. Suppose that there is a series class containing at least three elements e, f, g. In this case, $M \setminus e$ will have at least 2 coloops. But as M is paving all its minors are also paving. Thus, $M \setminus e$, being a paving matroid with at least 2 coloops, cannot have circuits and $M \setminus e$ is isomorphic to $U_{r,r}$. In this case, we conclude that M is isomorphic to $U_{r,r+1}$.

To finish, we suppose that every element in M is in a series class of size 2. In this case, M is the 2-stretching of a rank-s matroid N with m elements and $s \geq 1$. N is paving because it is a minor of M and it must have circuits as M is coloopless.

If the minimal size of a circuit in N has size s, M has a circuit of size $2s$. But the rank of M is $s + m$ as it is the 2-stretching of N. Then $2s \geq s + m$ and $s = m$. In this case, N would be isomorphic to $U_{s,s}$ and we arrive at a contradiction. Thus, N does not have circuits of size s.

Hence all the circuits of N have size $s + 1$ and N is uniform. Then, there is a circuit in M of size $2s + 2 \geq s + m$, and $s + 2 \geq m \geq s + 1$. Thus, N is isomorphic to $U_{s,s+1}$ or $U_{s,s+2}$. $\square$

Lemma 4.8. *Let M be a rank-r coloopless paving matroid. If for every element e of M, $M \setminus e$ has a coloop, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. We analyse the cases for M given in the previous lemma. If M is isomorphic to $U_{r,r+1}$, the result follows from Lemma 4.5. If M is isomorphic to $U_{1,2} \oplus U_{1,2}$ or $U_{1,2}$, the corresponding Tutte polynomials are $(x+y)^2$ and $x+y$, which in both cases are convex.

If M is the 2-stretching of $U_{s,s+1}$, then M is isomorphic to $U_{r,r+1}$ and the result follows from Lemma 4.5. If M is the 2-stretching of $U_{s,s+2}$, then M^* is the 2-thickening of $U_{2,n}$ which is a rank-2 matroid and the result follows from Theorem 4.6 and Lemma 4.2. $\square$

Finally, we arrive at the main result of this section.

Theorem 4.9. *If M is a coloopless paving matroid, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. If M has a loop, then M has rank 1 and it is isomorphic to $U_{1,k+1} \oplus U_{0,l}$ with $l, k \geq 1$ and the result follows from Lemma 4.4.

Otherwise, every element of M is neither a loop nor a coloop. If there is an element e such that $M \setminus e$ has no coloop, then both M/e and $M \setminus e$ are coloopless paving matroids and the result follows from Lemma 4.3.

So, we can assume that for all e, $M \setminus e$ has a coloop. Then the result follows from Lemma 4.8. □

Hence, subject to an affirmative answer to Welsh's problem mentioned earlier, we have proved Conjecture 1.1 and Theorem 4.9 for asymptotically almost all matroids.

Paving matroids are not closed under duality but using Lemma 4.2 we obtain the convexity of the Tutte polynomial for a bigger class of matroids.

Corollary 4.10. *If M or M^* is a coloopless paving matroid, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

By Theorem 4.1, the class of matroids M such that either M or M^* is a coloopless paving matroid is contained in the class of matroids that contains two disjoint bases or whose ground set is the union of two bases. Thus, we have a strengthening of Theorem 3.3.

Corollary 4.11. *If M or M^* is a coloopless paving matroid, then T_M satisfies inequality (11) for $a \geq 0$.*

5 The Merino-Welsh conjecture

In this section we return to the original Merino–Welsh conjecture (Conjecture 1.1) and establish that its conclusion holds for some fairly specific classes of graphs and matroids. Recall that the conclusion of the conjecture is certainly not true for all graphs. Taking any graph and adding a loop and a bridge results in a graph that does not satisfy (1). However, the condition on the connectivity may not be the most natural because if G consists of 2 cycles of length 2 sharing a common vertex, then the graphic matroid $M(G)$ satisfies (9) for all $a \geq 0$. So (1) is satisfied by some graphs that are not 2-connected.

5.1 Wheels and whirls

In this subsection we consider wheels, a well-known class of self-dual planar graphs, and whirls, a related class of matroids which are also self-dual. The wheel graph W_n has $n + 1$ vertices and $2n$ edges. The vertices $\{1, \cdots, n\}$ form an n-cycle while the vertex 0 is adjacent to every vertex in this cycle. The whirl

W^n is the matroid with ground set $E(W^n) = E(W_n)$, while the set of bases of W^n consists of the edge set in the n-cycle of W_n together with all edge sets of spanning trees of W_n, see [22].

It is well-known that $\tau(W_n) = L_{2n} - 2$, for $n \geq 1$, where L_k is the kth-Lucas number which is defined recursively by $L_1 = 1$, $L_2 = 3$ and $L_k = L_{k-1} + L_{k-2}$ for $k \geq 3$. This result was proved by Sedláček [23] and also by Myers [20]. Using the analogy of Binet's Fibonacci formula for Lucas numbers we get

$$\tau(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2.$$

The same formula can be obtained directly by using equation (4) for $T_{W_n}(1,1)$ and then solving the corresponding recurrence relation.

The chromatic polynomial of W_n is known, see [2], and is equal to $\chi_{W_n}(x) = x(x-2)^n + (-1)^n x(x-2)$. Now, applying the famous result of R. Stanley [24] that relates the number of acyclic orientations and the chromatic polynomial, namely $\alpha(G) = |\chi_G(-1)|$, we get $\alpha(W_n) = 3^n - 3$. These results together yield the following

Theorem 5.1. *For all $n \geq 2$, $\alpha(W_n) \geq \tau(W_n)$ and $M(W_n)$ satisfies Conjecture 1.1.*

The Tutte polynomials of W^n and $M(W_n)$ are related by the equality, $T_{W^n}(x,y) = T_{W_n}(x,y) - xy + x + y$. Thus, obtain the following result

Theorem 5.2. *For all $n \geq 2$, $T_{W^n}(2,0) \geq T_{W^n}(1,1)$ and W^n satisfies equation (1).*

5.2 3-regular graphs with girth at least 5

For 3-regular graphs with girth at least 5 a lower bound for the number of acyclic orientations,

$$\alpha(G) \geq (2^{3/8} 3^{3/8} 4^{1/8})^n ,$$

is given in [14], where n is the number of vertices of G. On the other hand, the following upper bound for the number of spanning trees in a 3-regular graph G is given in [7].

$$\tau(G) \leq \frac{2\beta}{3n} e^{\frac{12}{\sqrt{\pi}}\left(\frac{1}{\beta}\right)^{\frac{5}{2}}} \left(\frac{4}{\sqrt{3}}\right)^n ,$$

where $\beta = \lceil \ln(n)/\ln(9/8) \rceil$. From the formulae we obtain the following

Theorem 5.3. *If G is a 3-regular graph of girth at least 5, we have $\tau(G) < \alpha(G)$ and $M(G)$ satisfies Conjecture 1.1.*

5.3 Complete graphs

It is natural to check if Conjecture 1.1 is true for complete graphs and complete bipartite graphs.

A classical result of Cayley [2] states that $\tau(K_n) = n^{n-2}$. For K_3 we have $\alpha(K_3) = 6 > 3 = \tau(K_3)$, thus K_3 satisfies Conjecture 1.1.

We use the following lemma which has an easy proof, see [8].

Lemma 5.4. *If G is a 2-connected graph with a vertex v of degree d, then $(2^d - 2)\alpha^*(G - v) \leq \alpha^*(G)$.*

We will prove that $\alpha^*(K_n) \geq n^{n-2}$, for $n \geq 4$. When $n = 4$, we have $\alpha^*(K_4) = 24 > 16 = \tau(K_4)$. We proceed by induction on n.

$$\begin{aligned}
\tau(K_{n+1}) = (n+1)^{n-1} &= \left(\frac{n+1}{n}\right)^n \left(\frac{n}{n+1}\right)^2 (n+1)\tau(K_n) \\
&\leq e(n+1)\tau(K_n) \leq (2^n - 2)\tau(K_n) \\
&\leq (2^n - 2)\alpha^*(K_n).
\end{aligned}$$

The last quantity is less than or equal $\alpha^*(K_{n+1})$ by the previous lemma.

Theorem 5.5. *For all $n \geq 3$, $M(K_n)$ satisfies Conjecture 1.1.*

The technique used for complete graphs can be used to prove the Conjecture 1.1 in the case of threshold graphs, a type of chordal graphs, see [8]. Also in [8] complete bipartite graphs are considered and the authors prove the following

Theorem 5.6. *For all $m \geq n \geq 2$, $M(K_{n,m})$ satisfies Conjecture 1.1.*

5.4 Catalan matroids

A *Dyck path* of length $2n$ is a path in the plane from (0,0) to ($2n$,0), with steps (1,1), called *up-steps*, and (1,-1), called *down-steps*. It is well-known that the number of Dyck paths of length $2n$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Each Dyck path P defines an *up-step set*, consisting of the integers i, $1 \leq i \leq 2n$, for which the ith-step of P is an up-step. The collection of up-step sets of all Dyck paths of length $2n$ forms the bases of a matroid M_n over $\{1, 2, \ldots, 2n\}$. These matroids are called Catalan matroids and have recently been studied extensively, see [4] or [1].

We consider the matroids N_n, $n \geq 2$, obtained form M_n by deleting the elements 1 and $2n$. This corresponds to deleting the loop and isthmus of M_n. From the results in [4] it follows that the matroid N_n is self-dual, but not identically self-dual. An expression for the Tutte polynomial of N_n follows from Corollary 5.8 of [4].

$$T_{N_n}(x, y) = \sum_{i,j>0} \frac{i+j-2}{n-1} \binom{2n-i-j-1}{n-i-j+1} x^{i-1} y^{j-1}.$$

After some algebraic manipulations we get a formula for the evaluation at (2,0) and (0,2).

$$T_{N_n}(2,0) = T_{N_n}(0,2) = \sum_{k=0}^{m} \frac{k}{m} \binom{2m-k-1}{m-k} 2^k,$$

where $m = n-1$. This quantity equals $\binom{2m}{m}$ by the following list of equalities

$$\begin{aligned}
\sum_{k=0}^{m} \frac{k}{m} \binom{2m-k-1}{m-k} 2^k &= \sum_{k=0}^{m} \left(\binom{2m-k-1}{m-1} - \binom{2m-k-1}{m} \right) 2^k \\
&= \sum_{k=0}^{m} \sum_{j=0}^{k} \left(\binom{2m-k-1}{m-1} - \binom{2m-k-1}{m} \right) \binom{k}{j} \\
&= \sum_{j=0}^{m} \sum_{k=j}^{m} \left(\binom{2m-k-1}{m-1} - \binom{2m-k-1}{m} \right) \binom{k}{j} \\
&= \sum_{j=0}^{m} \left(\binom{2m}{m+j} - \binom{2m}{m+j+1} \right) \\
&= \binom{2m}{m}.
\end{aligned}$$

The key step in the middle uses the convolution identity $\sum_{k=0}^{2m-1} \binom{2m-k-1}{q} \binom{k}{j} = \binom{2m}{q+j+1}$ that is the basic identity (5.6) in [11]. The value of $T_{N_n}(1,1)$ is clearly $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem 5.7. *For all $n \geq 2$, N_n satisfies equation (1).*

Notice that in all of the classes that we have considered, either the ground set contains two disjoint bases or is the union of two bases. We therefore propose the following conjecture which is a weaker form of Conjecture 1.1 and may turn out to be more tractable.

Conjecture 5.8. *If M contains two disjoint bases or its ground set is the union of two bases then* $\max\{T_M(2,0),\ T_M(0,2)\} \geq T_M(1,1)$.

6 Conclusion and Discussion

We have proved that T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant, whenever M is a coloopless paving matroid. By Definition 2.3, T_M is convex along the semilines $y - mx + b$ for $m \geq 0$ and $b \in \mathbb{R}$ in the positive quadrant. It is natural to ask for which matroids is T_M convex in the positive quadrant?

There is no clear link between convexity of the Tutte polynomial in the positive quadrant and the classes of matroids that we have considered. Coloopless paving matroids may or may not have Tutte polynomials that are convex in

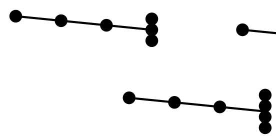

Figure 1

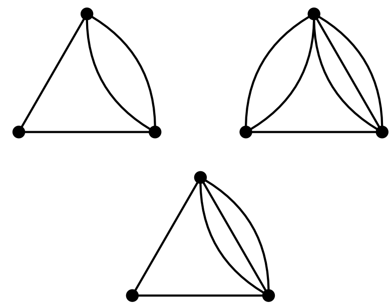

Figure 2

the positive quadrant. For example, the Tutte polynomials of uniform matroids and the graphic matroid $M(K_4)$ are convex in the positive quadrant; on the other hand the Tutte polynomial $y^l(y^k + \ldots + y + x)$ of the paving matroid $U_{1,k+1} \oplus U_{0,l}$, where $l \geq 1$ and $k \geq 1$ is not a convex or concave function. There are also non-paving matroids whose Tutte polynomial is convex, for example $U^2_{n,n}$, for $n \geq 3$, the 2-thickening of $U_{n,n}$. The Tutte polynomial of this matroid is $(x+y)^n$ which is clearly convex. Note however that this latter class of matroids has two disjoint bases.

Establishing the convexity of the Tutte polynomials of matroids within a given large class seems to be a difficult problem. The Tutte polynomials of the graphs at the top of Fig. 1 are convex functions while the Tutte polynomial of the graph at the bottom is neither convex nor concave. A similar situation holds for the matroids in Fig. 2, the Tutte polynomials of the two matroids at the top of the figure are convex functions while the polynomial for the matroid at the bottom is neither convex nor concave.

We proved Conjecture 1.1 for some families of graphs and matroids. There are some more families for which the conjecture holds: for example Marc Noy (private communication) proved that $\tau(G) \leq \alpha(G)$ when G is a maximal outerplanar graph, using equation (4).

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